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SIMPLE WAVES IN NONLINEARLY ELASTIC MEDIA*

E. I. SVESHNIKOVA

Plane isentropic waves (simple waves) are studied in the prestressed medium. Propagation of simple waves through a stress-free medium was studied by Bland in /1/. Below the variation in the parameters of simple waves in a prestressed medium is determined together with the characteristic velocities, and their dependence on the current state of the medium and on the prior deformation. It is shown that all three waves (one quasilongitudinal and two quasitransverse) can break, and conditions under which this can happen are given.

1. Formulation of the problem. A nonlinearly elastic isotropic medium is described by the function $\Phi = \rho_0 U(\varepsilon_{ij}, S)$ where U is the internal energy of the unit mass, ρ_0 is density in the stress-free state,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i} + \frac{\partial w_s}{\partial \xi_i}\frac{\partial w_s}{\partial \xi_j}\right)$$

are the deformation tensor components, w_i are the displacement vector components and S is entropy. The propagation of simple plane waves is considered in ξ_i the Lagrangian coordinate system, which becomes rectangular and Cartesian in the stress-free state medium. The equations of motion of elastic medium in Lagrangian variables are /1/

$$\rho_0 \frac{\partial^2 w_i}{\partial t^2} = \frac{\partial}{\partial \xi_j}\frac{\partial \Phi}{\partial(\partial w_i/\partial \xi_j)}, \quad i = 1, 2, 3 \tag{1.1}$$

and repeated indices denote summation.

We seek solutions of the system (1.1) of the form $w_i = w_{i0} + w_i^*(\theta(\xi_3, t))$, $S = \mathrm{const}$ and θ is a function of its arguments. In such a wave, out of all $\partial w_k/\partial \xi_i$ characterizing the deformation, only $\partial w_k/\partial \xi_3$ will vary, the rest $\partial w_k/\partial \xi_\alpha$, $\alpha = 1, 2$ remaining constant. The latter are determined by the initial deformation which will be assumed homogeneous. For the system (1.1) to have such solutions, it is necessary for the system of algebraic equations for $du_i/d\theta$ (where $u_i = \partial w_i/\partial \xi_3$)

$$\left(\frac{\partial^2 \Phi}{\partial u_i \partial u_j} - \rho_0 c^2 \delta_{ij}\right)\frac{du_j}{d\theta} = 0, \quad c = -\frac{\partial\theta/\partial t}{\partial\theta/\partial \xi_3}, \quad i = 1, 2, 3 \tag{1.2}$$

to have nontrivial solutions. Clearly, $c = d\xi_3/dt$ is the rate of displacement of the surface $\theta(\xi_3, t) = \mathrm{const}$ with respect to variable ξ_3, i.e. it is the characteristic velocity. We see that the quantities $du_i/d\theta$ are represented by the eigenvector of the matrix $\|\partial^2\Phi/(\partial u_i \partial u_j)\|$ and $\alpha = \rho_0 c^2$ are the eigenvalues of this matrix. Every eigenvalue corresponds to two identical waves propagating into the opposite sides of the ξ_3 axis.

We shall consider simple waves in the domain of small ε_{ij}. In /1/ the author investigated in detail the simple waves for a prestressed state and for a particular type of initial deformations, namely such deformations that $\partial w_1/\partial \xi_1 = \partial w_2/\partial \xi_2$, this ensuring the isotropic character of the deformations in the planes parallel to the wave front. The function Φ in this case depends on two variables only, u_3 and $\sqrt{u_1^2 + u_2^2}$. Below we shall consider the deformations of arbitrary form, but small enough, so that their squares will be much smaller than unity and therefore neglected. The function Φ will be given for convenience, in the form of an expansion in powers of ε_{ij}.

$$\Phi = {}^1/_2\lambda I_1^2 + \mu I_2 + \beta I_1 I_2 + \gamma I_3 + \delta I_1^3 + \xi I_2^2 + \eta I_1 I_3 + \zeta I_1^2 I_2 + \omega I_1^4 \tag{1.3}$$

$$I_1 = \varepsilon_{kk}, \quad I_2 = \varepsilon_{ik}\varepsilon_{ik}, \quad I_3 = \varepsilon_{ik}\varepsilon_{kj}\varepsilon_{ji}$$

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Let us set $\partial w_3/\partial\xi_1 = \partial w_3/\partial\xi_2 = 0$. Then the smallness of the deformations will imply the smallness of u_i. The expansion of Φ in terms of u_i contains all three variables, with different coefficients, which are functions of the initial deformations. The directions of the axes ξ_1, ξ_2 are chosen so that $\partial w_1/\partial\xi_2 + \partial w_2/\partial\xi_1 = 0$. This enables us to assume, within the adopted approximation to the initial deformations, that $\varepsilon_{12} = 0$. Thus the initial deformation will be defined by two compression (tension) components $\varepsilon_{11}, \varepsilon_{22}$ which remain constant within the wave, and by the initial values $u_i = u_i^0$.

In a weakly deformed medium defined by the potential (1.3), the coefficients of the equations (1.1) have the following form with the accuracy of up to the second order infinitesimals:

$$\begin{aligned}
&\partial^2\Phi/\partial u_1\partial u_1 \approx f_{11} = f + f_1 u_3 + h\,(3u_1^2 + u_2^2) + f_2\cdot u_3^2 \\
&\partial^2\Phi/\partial u_2\partial u_2 \approx f_{22} = g + g_1 u_3 + h\,(u_1^2 + 3u_2^2) + g_2 u_3^2 \\
&\partial^2\Phi/\partial u_3\partial u_3 \approx f_{33} = d + d_1 u_3 + d_2\,(u_1^2 + u_2^2) + d_3 u_3^2 \\
&\partial^2\Phi/\partial u_1\partial u_3 \approx f_{13} = f_1 u_1, \quad \partial^2\Phi/\partial u_2\partial u_3 \approx f_{23} = g_1 u_2 \\
&\partial^2\Phi/\partial u_1\partial u_2 \approx f_{12} = 2hu_1u_2 \\
&d = \lambda + 2\mu + O(\varepsilon), \quad f = \mu + O(\varepsilon), \quad g = \mu + O(\varepsilon) \\
&d_1 = 6a + O(\varepsilon), \quad f_1 = 2b + O(\varepsilon), \quad g_1 = 2b + O(\varepsilon) \\
&a = \lambda/2 + \mu + \beta + \gamma + \delta, \quad 2b = \lambda + 2\mu + \beta + {}^3/_2\gamma \\
&h = \lambda/2 + \mu + \beta + {}^3/_2\gamma + \xi
\end{aligned} \tag{1.4}$$

Here $O(\varepsilon)$ represent terms of the order of initial deformation components ε_{11} and ε_{22}. We shall see later that the coefficients d_2, d_3, f_2 and g_2 do not appear in the solution of the problem. The following cubic equation is used to determine the eigenvalues α

$$\det \| f_{ij} - \alpha\delta_{ij} \| = 0 \tag{1.5}$$

In the case of a stress-free state (all $\varepsilon_{ij} = 0$) the above equation yields three known roots: $\alpha_{1,2} = \mu$ for the two transvers waves and $\alpha_3 = \lambda + 2\mu$ for the longitudinal wave. For a weakly deformed initial state (small u_i and $\varepsilon_{11}, \varepsilon_{22}$) the roots of (1.5) can be obtained approximately by computing a small correction to the values given above. The first (principal) term of this correction will suffice here.

2. Quasilongitudinal wave.

Computing the first root of (1.5) we obtain

$$\alpha_3 = \rho_0 c_3^2 = \lambda + 2\mu + (\lambda + 2\beta + 6\delta)(\varepsilon_{11} + \varepsilon_{22}) + 6au_3$$

This shows that the characteristic velocity depends on the deformation of the medium. Since in the nontrivial solution $du_3 \neq 0$, i.e. u_3 varies within the wave, it follows that $c_3 \neq$ const and the wave will show the tendency to break. For the materials where $a > 0$, the rarefaction waves, in which u_3 is increasing, will break, and for the materials with $a < 0$ the compression waves break.

The eigenvector corresponding to α_3 is found from the system (1.2) using (1.4), and has the form

$$\frac{du_k}{du_3} = \frac{2bu_k}{\lambda + \mu + 2(3a - b)u_3}, \qquad k = 1, 2$$

Integrating this gives

$$u_k = u_k^\circ\left(1 + \frac{2(3a - b)(u_3 - u_3^\circ)}{\lambda + \mu}\right)^q, \qquad q = \frac{b}{3a - b}$$

If the initial deformation is absent, the wave will be purely longitudinal. If on the other hand $u_k^\circ \neq 0$, then a small transverse component will appear proportional to the small initial shear deformation and to the change in the longitudinal componet u_3. Such a wave can be called quasilongitudinal.

3. Quasitransverse waves.

In the course of computing the approximate values of the other two roots of (1.5) we found that the dependence of the characteristic velocities on the actual state of the medium manifests itself in the terms beginning with u_i^2. The third equation of the system (1.2) and the coefficients (1.4) together yield a relation connecting the longitudinal and transverse components of the deformation, in the wave which is almost transverse, using the first approximation for the characteristic velocity, i.e. assuming that $\alpha = \mu$. When the terms of second order of smallness are retained, we obtain, after integrating,

$$u_3 = u_3^\circ - b\,(u_1^2 + u_2^2)/(\lambda + \mu) \tag{3.1}$$

This shows that the variation in the longitudinal component is smaller than that of the transverse component by one order of magnitude. We shall call such wave quasitransverse.

The characteristic velocities of these waves are obtained from the first and second equations of (1.2) using (1.4) and the expression (3.1) for u_3. The formulas for the characteristic velocities of these waves have the form

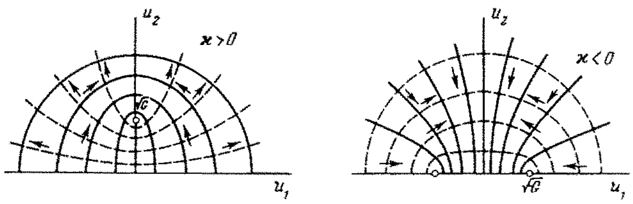
$$\alpha_{1,2} = \rho_0 c_{1,2}^2 = (\alpha_1^\circ + \alpha_2^\circ)/2 - \varkappa \{u_1^2 + u_2^2 \pm {}^1/_2 [(u_2^2 - u_1^2 - G)^2 + 4u_1^2 u_2^2]^{1/2}\}$$
$$\alpha_i^\circ = \mu + 2bI_1^\circ - (2\mu + {}^3/_2\gamma)\,\varepsilon_{ii}, \quad i = 1, 2$$
$$\varkappa = \mu + (\mu + \beta + {}^3/_2\gamma)^2/(\lambda + \mu) - 2\xi, \quad G = (\alpha_1^\circ - \alpha_2^\circ)/\varkappa \tag{3.2}$$

Here α_1° and α_2° denote the values of α for the waves propagating along a medium in a state in which $u_1 = u_2 = 0$. The values depend on the initial deformations of the medium $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}^\circ$ which remains unchanged within the wave, and they differ from each other, albeit by a small amount. The formulas for c_i^2, $i = 1, 2, 3$ in a prestressed medium are also given in /2/.

Using appropriate numbering for the coordinate axes, we can always make $\alpha_1^\circ - \alpha_2^\circ > 0$. Then the sign of G and of the coefficient $\varkappa$ specified by the elastic properties of the medium, will be the same. We have already said, and now it becomes apparent from the formulas (3.2) that the dependence of the characteristic velocities c_1 and c_2 and u_i manifests itself only in the quadratic terms. The upper sign in the formulas (3.2) yields, for $\varkappa > 0$, the characteristic velocity of the wave which can be called a "fast" quasitransverse wave, and the lower sign corresponds to the "slow" quasitransverse wave. The relations are reversed for $\varkappa < 0$. Both waves have a tendency to break, but the fact can be explained, when it happens, only when the change in the solution itself is known for each wave.

4. Integral curves for the quasitransverse simple waves. The differential equation for finding the integral curves depicting a simple wave on the u_1, u_2-plane is obtained from the last equations of (1.2), using (1.4) and (3.2)

$$\frac{du_2}{du_1} = \frac{u_2^2 - u_1^2 - G \mp [(u_2^2 - u_1^2 - G)^2 + 4u_1^2u_2^2]^{1/2}}{2u_1u_2} \tag{4.1}$$



From (4.1) we see that we have two, mutually orthogonal families of integral curves depicting two quasitransverse waves propagating with velocities c_1 and c_2 respectively. The distribution of the $\pm$ signs in the formulas (3.2) and (4.1) corresponds to each other. The curves (4.1) have two singularities with coordinates $u_1 = 0$, $u_2 = \pm\sqrt{G}$ for the materials with $\varkappa > 0$ $(G > 0)$, and coordinates $u_1 = \pm\sqrt{-G}$, $u_2 = 0$ for the media with $\varkappa < 0$ $(G < 0)$. In the left-hand part of the above figure for the material with $\varkappa > 0$ the solid lines depict the curves belonging to the first family corresponding to the upper sign in (4.1) and (3.2) (fast waves). The lines of the second family (slow waves) are depicted by the dashed lines. On moving away from the singularities when $u_i \gg \sqrt{G}$, the curves of the first family tend to circles, and those of the second family to rays. Similarly, in the right-hand part of the figure the integral curves are depicted for a material with $\varkappa < 0$. The solid lines correspond to the fast waves, and the dashed lines to the slow waves. Both patterns of the integral curves and symmetrical about the u_1, u_2 axes. The condition $u_i \gg \sqrt{|G|}$ means that the initial deformation is such that the difference $\varepsilon_{11} - \varepsilon_{22}$ is small compared with u_i^2. In the limit when $\varepsilon_{11} - \varepsilon_{22} = 0$, all integral curves become circles and rays, and the case is described in /1/. We see that the integral curves for the media with $\varkappa > 0$ and $\varkappa < 0$ can be obtained from each other by rotating the axes by $\pi/2$.

5. Breaking of the quasitransverse simple waves. In order to find out which waves show the tendency to break, we compute the derivatives fo the characteristic velocities along their integral curves. Along the curve represented by the equation $u_2 = u_2(u_1)$ the characteristic velocities are written in the form $c_i = c_i(u_1, u_2(u_1))$. The derivatives along this line can be found as follows:

$$\frac{dc_i}{du_1}=\frac{\partial c_i}{\partial u_1}+\frac{\partial c_i}{\partial u_2}\frac{du_2}{du_1}=-\frac{3}{4}\frac{\varkappa}{\rho_0 c_i}\frac{u_1^2+u_2^2-G\mp\Delta}{\Delta} \tag{5.1}$$
$$\Delta=[(u_2^2-u_1^2-G)^2+4u_1^2u_2^2]^{1/2},\quad i=1,2$$

When $\varkappa>0$, the fast waves in which $|u_1|$ increases and the slow waves in which $|u_1|$ decreases may both break. When $\varkappa<0$, both fast and slow waves in which $|u_1|$ increases show the tendency to break. In the figure the arrows indicate the directions of the parameter changes which correspond to the nonbreaking simple waves.

The results given here can be obtained by a different method, using the arguments of /3/ and considering very low intensity shock waves. A simple wave can be regarded as a set of infinitely weak shock waves, each of which propagates through a medium deformed by the passage of the previous shock waves. The integral curves can be then constructed for simple waves from the segments of the initial directions of the shock adiabates for each state. We can see that the direction of increasing entropy on the segments of the shock adiabate coincides with the directions of the increasing characteristic velocities along the integral curves of the simple waves. When $\varepsilon_{11}-\varepsilon_{22}=0$ $(G=0)$, a part of the shock adiabate /3/ becomes a circle coinciding with the circle of constant entropy. In this case a nonbreaking simple wave and a discontinuity propagating with a constant velocity $\alpha=(\alpha_1^\circ+\alpha_2^\circ)/2-{}^1/_2\,\varkappa\,(u_1^2+u_2^2)+2bu_3^\circ$ can both exist without changing the form.

It should be noted that since the expansion in small deformations of the initial state was continued as in /3/ only up to the linear terms (in the present paper to the overall second power), the coefficients of the adiabate differ slightly from those of the expressions for the velocity. They will coincide when the term $2kh\gamma z\delta/b^2$ is added to the formula (5.1) of /3/, and this does not alter the qualitative results of /3/.

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